Stopping Games in Continuous Time*

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Abstract

We study two-player zero-sum stopping games in continuous time and infinite horizon. We prove that the value in randomized stopping times exists as soon as the payoff processes are right-continuous. In particular, as opposed to existing literature, we do *not* assume any conditions on the relations between the payoff processes. We also show that both players have simple ε-optimal randomized stopping times; namely, randomized stopping times which are small perturbations of non-randomized stopping times.

Keywords: Dynkin games, stopping games, optimal stopping, stochastic analysis, continuous time.

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1 Introduction

Stopping games in discrete time were introduced by Dynkin (1969) as a variation of optimal stopping problems. In Dynkin's (1969) setup, two players observe the realization of two discrete time processes $(x_t, r_t)_{t\in\mathbf{N}}$. Player 1 chooses a stopping time μ such that $\{\mu = t\} \subseteq \{r_t \geq 0\}$ for every $t \in \mathbf{N}$, and player 2 chooses a stopping time ν such that $\{\nu = t\} \subseteq \{r_t < 0\}$ for every $t \in \mathbf{N}$. Thus, players are not allowed to stop simultaneously. Player 2 then pays player 1 the amount $x_{\min\{\mu,\nu\}}\mathbf{1}_{\min\{\mu,\nu\}<+\infty}$, where $\mathbf{1}$ is the indicator function. This amount is a random variable. Denote the *expected payoff* player 1 receives by

$$\gamma(\mu,\nu) = \mathbf{E}[x_{\min\{\mu,\nu\}}\mathbf{1}_{\min\{\mu,\nu\}<+\infty}].$$

Dynkin (1969) proved that the game admits a value; that is,

$$\sup_{\mu}\inf_{\nu}\gamma(\mu,\nu) = \inf_{\nu}\sup_{\mu}\gamma(\mu,\nu).$$

Since then many authors generalized this basic result, both in discrete time and in continuous time.

In *discrete time*, Neveu (1975) allows the players to stop simultaneously, that is, he introduces three uniformly integrable adapted processes $(a_t, b_t, c_t)_{t\in\mathbf{N}}$, the two players choose stopping times μ and ν respectively, and the payoff player 2 pays player 1 is

$$a_\mu \mathbf{1}_{\{\mu<\nu\}} + b_\nu \mathbf{1}_{\{\mu>\nu\}} + c_\mu \mathbf{1}_{\{\mu=\nu<+\infty\}}.$$

Neveu (1975) provides sufficient conditions for the existence of the value. One of the conditions he imposes is the following:

- **Condition C**: $c_t = a_t \leq b_t$ for every $t \geq 0$.

It is well known that in general the value need not exist when condition C is not satisfied. Rosenberg et al. (2001) allow the players to choose *randomized* stopping times, and they prove, in discrete time again, the existence of the value in randomized stopping times. This result was recently

generalized by Shmaya and Solan (2002) to the existence of an ε-equilibrium in the non-zero-sum problem.

Several authors, including Bismut (1979), Alario-Nazaret et al. (1982) and Lepeltier and Maingueneau (1984) studied the problem in *continuous time*. That is, the processes $(a_t, b_t, c_t)_{t \geq 0}$ are in continuous time, and the stopping times the players choose are $[0, +\infty]$-valued. The literature provides sufficient conditions, that include condition C, for the existence of the value in pure (i.e. non-randomized) stopping times.

Touzi and Vieille (2002) study the problem in continuous time, without condition C, played on a bounded interval $[0, T]$; that is, players must stop before or at time T. They prove that if $(a_t)_{t \geq 0}$ and $(b_t)_{t \geq 0}$ are semimartingales continuous at T, and if $c_t \leq b_t$ for every $t \in [0, T]$, then the game admits a value in randomized stopping times.

In the present paper we prove that every stopping game in continuous time where $(a_t)_{t \geq 0}$ and $(b_t)_{t \geq 0}$ are right-continuous, and $(c_t)_{t \geq 0}$ is progressively measurable, admits a value in randomized stopping times. In addition, we construct ε-optimal strategies which are as close as one wishes to pure (non-randomized) stopping times; roughly speaking, there is a stopping time μ such that for every δ sufficiently small there is an ε-optimal strategy that stops with probability 1 between times μ and $\mu + \delta$. Finally, we construct an ε-optimal strategy in the spirit of Dynkin (1969) and we extend the model by introducing cumulative payoffs and final payoffs.

Stopping games in continuous time were applied in various contexts. The one player stopping problem (the Snell envelope) is used in finance for the pricing of the American option, see, e.g., Bensoussan (1984) and Karatzas (1988). More recently Cvitanic and Karatzas (1996) used stopping games for the study of backward stochastic differential equation with reflecting barriers, and Ma and Cvitanic (2001) for the pricing of "the American game option". Ghemawat and Nalebuff (1985) used stopping games to study strategic exit from a shrinking market.

2 Model, literature and main result

A *two-player zero-sum stopping game in continuous time* Γ is given by:

- A probability space $(\Omega, \mathcal{A}, P)$: $(\Omega, \mathcal{A})$ is a measurable space and P is a σ-additive probability measure on $(\Omega, \mathcal{A})$.

- A filtration in continuous time $\mathcal{F} = (\mathcal{F}_t)_{t \geq 0}$ satisfying "the usual conditions". That is, $\mathcal{F}$ is right-continuous, and $\mathcal{F}_0$ contains all P-null sets: for every $B \in \mathcal{A}$ with $P(B) = 0$ and every $A \subset B$, one has $A \in \mathcal{F}_0$.

  Denote $\mathcal{F}_\infty := \vee_{t \geq 0} \mathcal{F}_t$. Assume without loss of generality that $\mathcal{F}_\infty = \mathcal{A}$. Hence $(\Omega, \mathcal{A}, P)$ is a complete probability space.

- Three uniformly bounded $\mathcal{F}$-adapted processes $(a_t, b_t, c_t)_{t \geq 0}$.[1]

A *pure strategy* of player 1 (resp. player 2) is a $\mathcal{F}$-adapted stopping time μ (resp. ν). We allow players to never stop, by choosing μ (or ν) to be equal to $+\infty$.

The game proceeds as follows. Player 1 chooses a pure strategy μ, and player 2 chooses simultaneously and independently a pure strategy ν. Player 2 then pays player 1 the amount $a_\mu \mathbf{1}_{\{\mu < \nu\}} + b_\nu \mathbf{1}_{\{\mu > \nu\}} + c_\mu \mathbf{1}_{\{\mu = \nu < +\infty\}}$, which is a random variable. The *expected payoff* that correspond to a pair of pure strategies (μ, ν) is

$$\gamma(\mu, \nu) = \mathbf{E}_P[a_\mu \mathbf{1}_{\{\mu < \nu\}} + b_\nu \mathbf{1}_{\{\mu > \nu\}} + c_\mu \mathbf{1}_{\{\mu = \nu < +\infty\}}].$$

Thus, if the game never stops, the payoff is 0. This could be relaxed by adding to the payoff a final payoff $\chi \mathbf{1}_{\mu = \nu = +\infty}$, where χ is some $\mathcal{A}$-measurable function; see Section 4. For a given stopping game Γ we denote the expected payoff by $\gamma_\Gamma(\mu, \nu)$ when we want to emphasize the dependency of the expected payoff on the game.

[1] As we argue below (see Section 2.1) our results hold for a larger class of payoff processes, that contains the class of uniformly integrable payoff processes.

The quantity $\sup_\mu \inf_\nu \gamma(\mu, \nu)$ is the maximal amount that player 1 can guarantee to receive; that is, the best he can get (in expectation) if player 2 knows the strategy chosen by player 1 before he has to choose his own strategy. Similarly, by playing properly, player 2 can guarantee to pay no more than $\inf_\nu \sup_\mu \gamma(\mu, \nu)$.

Definition 1 *If* $\sup_\mu \inf_\nu \gamma(\mu, \nu) = \inf_\nu \sup_\mu \gamma(\mu, \nu)$ *then the common value is the* value in pure strategies *of the game, and is denoted by* v*. Any strategy* μ *for which* $\inf_\nu \gamma(\mu, \nu)$ *is within* ε *of* v *is* an ε-optimal *strategy of player 1.* ε*-optimal strategies of player 2 are defined analogously.*

Many authors provided sufficient conditions for the existence of the value in pure strategies and ε-optimal pure strategies. The most general set of sufficient conditions in continuous time was given by Lepeltier and Maingueneau (1984, Corollary 12, Theorems 13 and 15).

Theorem 2 (Lepeltier and Maingueneau, 1984) *If (a) the processes* $(a_t, b_t)_{t \geq 0}$ *are right-continuous, and (b)* $a_t = c_t \leq b_t$ *for every* $t \geq 0$*, the value exists and both players have pure* ε*-optimal strategies.*

Remark 1: Lepeltier and Maingueneau (1984) require that the processes $(a_t, b_t)_{t \geq 0}$ are optional; that is, measurable with respect to the optional filtration. Recall that the optional filtration is the one generated by all RCLL (right-continuous with left limit) processes. Under the "usual conditions" it is also the filtration generated by all right-continuous processes (see, e.g., Dellacherie and Meyer, 1975, §IV-65).

Laraki (2000, Theorem 9.1) slightly extended this result by requiring that c_t is in the convex hull of a_t and b_t ($c_t \in \text{co}\{a_t, b_t\}$) for every $t \geq 0$ instead of (b) of Theorem 2.

The pure ε-optimal strategies that exist by Theorem 2 need not be finite. Indeed, the value of the game that is given by $a_t = c_t = -1$ and $b_t = 1$ for every $t \geq 0$ is 0, and the only 0-optimal pure strategy of player 1 is $\mu = +\infty$. Moreover, if μ is an ε-optimal pure strategy of player 1 then $P(\mu < +\infty) \leq \varepsilon$.

It is well known that in general the value in pure strategies need not exist. Indeed, take $a_t = b_t = 1$ and $c_t = 0$ for every $t \geq 0$. Since $\gamma(\mu, \mu) = 0$ it follows that $\sup_\mu \inf_\nu \gamma(\mu, \nu) = 0$. For every stopping time ν define a stopping time μ_ν by

$$\mu_\nu \begin{cases} 0 & \nu > 0 \\ 1 & \nu = 0 \end{cases} .$$

Since $\gamma(\mu_\nu, \nu) = 1$ for every ν it follows that $\inf_\nu \sup_\mu \gamma(\mu, \nu) = 1$, and the value in pure strategies does not exist.

The difficulty with the last example is that player 2, knowing the strategy of player 1, can stop exactly at the same time as his opponent. The solution is to allow player 1 to choose his stopping time randomly, thereby making the probability that the players stop simultaneously vanish. Indeed, in the last example, if player 1 could have randomly chosen his stopping time, say, uniformly in the interval $[0, 1]$, then the game terminates before time 1 with probability 1, and the probability of simultaneous stopping is 0, whatever player 2 plays. In particular, such a strategy guarantees player 1 payoff 1.

In the game theoretic literature, a standard and natural way to increase the set of strategies is by allowing players to randomize. A *mixed strategy* is a probability distribution over pure strategies. In general, this allows to convexify the set of strategies, and makes the payoff function bilinear. One can then apply a standard min-max theorem (e.g., Sion, 1958) to prove the existence of the value in mixed strategies, provided some regularity conditions hold (e.g., the space of mixed strategies is compact and the payoff function continuous).

Three equivalent ways to randomize the set of pure strategies in our setup are discussed in Touzi and Vieille (2002). We adopt the following definition of mixed strategies due to Aumann (1964). It extends the probability space to $([0, 1] \times [0, 1] \times \Omega, \mathcal{B} \times \mathcal{B} \times \mathcal{A}, \lambda \otimes \lambda \otimes P)$, where $\mathcal{B}$ is the σ-algebra of Borel sets of $[0, 1]$, and λ is the Lebesgue measure on $[0, 1]$.

Definition 3 *A* mixed strategy *of player 1 is a measurable function* $\phi : [0, 1] \times \Omega \to [0, +\infty]$ *such that for* λ*-almost every* $r \in [0, 1]$, $\mu_r(\omega) := \phi(r, \omega)$

is a stopping time.

The interpretation is the following: player 1 chooses randomly $r \in [0,1]$, and then stops the game at time $\mu_r = \phi(r,\cdot)$. Mixed strategies of player 2 are denoted by ψ, and, for every $s \in [0,1]$, the s-section is denoted by $\nu_s := \psi(s,\cdot)$.

The *expected payoff* that corresponds to a pair of mixed strategies (ϕ,ψ) is:

$$\begin{aligned} \gamma(\phi,\psi) &= \int_{[0,1]^2} \gamma(\mu_r,\nu_s)\ dr\ ds \qquad (1) \\ &= \mathbf{E}_{\lambda\otimes\lambda\otimes P}\left[a_{\mu_r}\mathbf{1}_{\{\mu_r<\nu_s\}} + b_{\nu_s}\mathbf{1}_{\{\mu_r>\nu_s\}} + c_{\mu_r}\mathbf{1}_{\{\mu_r=\nu_s<+\infty\}}\right]. \end{aligned}$$

Though the payoff function given by (1) is bilinear, without strong assumptions on the data of the game the payoff function is not continuous for the same topology which makes the strategy space compact.

Definition 4 *If* $\sup_\phi \inf_\psi \gamma(\phi,\psi) = \inf_\psi \sup_\phi \gamma(\phi,\psi)$ *then the common value is the* value in mixed strategies*, and it is denoted by* V*. Every strategy* ϕ *such that* $\inf_\psi \gamma(\phi,\psi)$ *is within* ε *of* V *is* ε-optimal *for player 1.* ε*-optimal strategies of player 2 are defined analogously.*

Observe that $\sup_\phi \inf_\psi \gamma(\phi,\psi) = \sup_\phi \inf_\nu \gamma(\phi,\nu)$, where ν ranges over all pure stopping times, and that $\inf_\psi \sup_\phi \gamma(\phi,\psi) = \inf_\psi \sup_\mu \gamma(\mu,\psi)$, where μ ranges over all pure stopping times. Hence, to prove the existence of the value, it suffices to show that $\sup_\phi \inf_\nu \gamma(\phi,\nu) = \inf_\psi \sup_\mu \gamma(\mu,\psi)$. Moreover, one always has $\sup_\phi \inf_\psi \gamma(\phi,\psi) \leq \inf_\psi \sup_\phi \gamma(\phi,\psi)$.

Existence of the value in mixed strategies in stopping games with continuous time was studied by Touzi and Vieille (2002), who proved the following.

Let Φ_T be the space of all mixed strategies ϕ such that $\lambda\otimes P(\mu_r \leq T) = 1$, and let Ψ_T be the space of all mixed strategies ψ such that $\lambda\otimes P(\nu_s \leq T) = 1$.

Theorem 5 (Touzi and Vieille, 2002) *For every* $T > 0$*, if (a) the processes* $(a_t,b_t)_{t\geq 0}$ *are semimartingales with trajectories continuous at time*

T, (b) $c_t \leq b_t$ for every $t \geq 0$, and (c) the payoff processes are uniformly integrable, then:

$$\sup_{\phi \in \Phi_T} \inf_{\psi \in \Psi_T} \gamma(\phi, \psi) = \inf_{\psi \in \Psi_T} \sup_{\phi \in \Phi_T} \gamma(\phi, \psi).$$

Touzi and Vieille (2002) prove that under conditions (a) and (b) of Theorem 5 it is sufficient to restrict the players to certain subclasses of mixed strategies. They then apply Sion's (1958) min-max theorem to the restricted game.

Remark 2: By Dellacherie and Meyer (1980, §VII-23), under the "usual conditions", a semimartingle is always RCLL (right-continuous with left limit).

One class of mixed strategies will play a special role along the paper.

Definition 6 *Let $\delta > 0$. A mixed strategy ϕ is δ-*almost pure *if there exists a stopping time μ and a set $A \in \mathcal{F}_\mu$ such that for every $r \in [0,1]$, $\phi(r, \cdot) = \mu$ on A, and $\phi(r, \cdot) = \mu + r\delta$ on A^c.*

Recall that a process $(x_t)_{t \geq 0}$ is progressively measurable if for every $t \geq 0$ the function $(s, \omega) \mapsto x_s(\omega)$ from $[0,t] \times \Omega$ is measurable with respect to $\mathcal{B}([0,t]) \times \mathcal{F}_t$, where $\mathcal{B}([0,t])$ is the σ-algebra of Borel subsets of $[0,t]$. Recall also that an optional process is progressively measurable (see, e.g., Dellacherie and Meyer, 1975, §IV-64).

The main result we present is the following.

Theorem 7 *If the processes $(a_t)_{t \geq 0}$ and $(b_t)_{t \geq 0}$ are right-continuous and if $(c_t)_{t \geq 0}$ is progressively measurable then the value in mixed strategies exists. Moreover, for every $\varepsilon > 0$ there is $\delta_0 \in (0,1)$ such that for every $\delta \in (0, \delta_0)$ both players have δ-almost pure ε-optimal strategies.*

Our proof heavily relies on the result of Lepeltier and Maingueneau (1984), where they extend the discrete time variational approach of Neveu (1975) to continuous time.

2.1 On the payoff processes

A $\mathcal{F}$-adapted process $x = (x_t)_{t\geq 0}$ is *in the class* $\mathcal{D}$ (see, e.g., Dellacherie and Meyer, 1980, §VI-20) if the set $\{x_\sigma \mathbf{1}_{\{\sigma<+\infty\}}, \sigma$ is a $\mathcal{F}$-adapted stopping time$\}$ is uniformly integrable (see, e.g., Dellacherie and Meyer, 1975, §II-17). That is, if for every bounded stopping time σ, $\mathbf{E}_P[|x_\sigma|\mathbf{1}_{\{x_\sigma \geq r\}}]$ converges uniformly to 0 as r goes to $+\infty$.

This implies that the set $\{\mathbf{E}_P[|x_\sigma|\mathbf{1}_{\{\sigma<+\infty\}}], \sigma$ is a $\mathcal{F}$-adapted stopping time$\}$ is uniformly bounded (see, e.g., Dellacherie and Meyer, 1975, §II-19). Observe that every uniformly bounded process, as well as every uniformly integrable process, is in the class $\mathcal{D}$ (see, e.g., Dellacherie and Meyer, 1975, §II-18)).

For a measurable process $(x_t)_{t\geq 0}$ and $r \geq 0$, define the process $(x_t^r)_{t\geq 0}$ by:

$$x_t^r(\omega) := x_t(\omega)\mathbf{1}_{\{|x_t(\omega)|\leq r\}} + r\mathbf{1}_{\{x_t(\omega)>r\}} - r\mathbf{1}_{\{x_t(\omega)<-r\}}.$$

By Dellacherie and Meyer (1975, §II-17), $x \in \mathcal{D}$ if and only if for every $\varepsilon > 0$ there exists $r > 0$ such that for every stopping time σ one has $E\left[|x_\sigma - x_\sigma^r|\,\mathbf{1}_{\{\sigma<+\infty\}}\right] < \varepsilon$. The process $(x_t^r)_{t\geq 0}$ is uniformly bounded by r, and, in addition, if $(x_t)_{t\geq 0}$ is right-continuous or $\mathcal{F}$-adapted, so is $(x_t^r)_{t\geq 0}$.

If the payoff processes $(a_t)_{t\geq 0}$, $(b_t)_{t\geq 0}$ and $(c_t)_{t\geq 0}$ are not necessarily bounded but are in the class $\mathcal{D}$, then for every $\varepsilon > 0$ there exists $r > 0$ such that

$$\mathbf{E}_P\left[(|a_\sigma - a_\sigma^r| + |b_\sigma - b_\sigma^r| + |c_\sigma - c_\sigma^r|)\,\mathbf{1}_{\{\sigma<+\infty\}}\right] < \varepsilon.$$

Hence, if the game $\Gamma = (\Omega, \mathcal{A}, P; \mathcal{F}, (a_t, b_t, c_t)_{t\geq 0})$ satisfies the assumptions of Theorem 7, it admits a value. Moreover, every ε-optimal strategy in $\Gamma^r := (\Omega, \mathcal{A}, P; \mathcal{F}, (a_t^r, b_t^r, c_t^r)_{t\geq 0})$ is 2ε-optimal in Γ.

In particular, all the existence results that are proved for uniformly bounded payoff processes (Lepeltier and Maingueneau (1984)) or uniformly integrable payoff processes (Touzi and Vieille (2002)) are valid for payoff processes in the class $\mathcal{D}$ as well.

3 Proof

In the present section the main result of the paper is proven. From now on we fix a stopping game Γ such that $(a_t, b_t)_{t\geq 0}$ are right-continuous and $(c_t)_{t\geq 0}$ is progressively measurable.

3.1 Preliminaries

The following Lemma will be used in the sequel.

Lemma 8 *For every $\mathcal{F}$-adapted stopping time τ and every $\varepsilon > 0$ there is $\delta > 0$ such that $p\left(\{|a_t - a_\tau| < \varepsilon \quad \forall t \in [\tau, \tau+\delta]\}\right) > 1 - \varepsilon$.*

A similar statement holds when one replaces the process $(a_t)_{t\geq 0}$ by the process $(b_t)_{t\geq 0}$.

Proof. Since $(a_t)_{t\geq 0}$ is right-continuous, it is progressively measurable (see, e.g., Dellacherie and Meyer, 1975, §IV-15).

Let $\delta_{\tau(\omega)}(\omega) = \inf\{s \geq \tau(\omega) : |a_s(\omega) - a_\tau(\omega)| \geq \varepsilon\}$. The progressive measurability of $(a_t)_{t\geq 0}$ implies that $\delta_{\tau(\omega)}(\omega)$ is measurable with respect to $\mathcal{F}_\infty$ (see, e.g., Dellacherie and Meyer, 1975, §III-44).

The right-continuity of $(a_t)_{t\geq 0}$ implies that $P(\{\omega : \delta_{\tau(\omega)}(\omega) > 0\}) = 1$. Since P is σ-additive and $\{\omega : \delta_{\tau(\omega)}(\omega) > 0\} = \cup_{n>0}\{\omega : \delta_{\tau(\omega)}(\omega) > \frac{1}{n}\}$, the lemma follows by choosing $\delta > 0$ sufficiently small so that $P(\{\omega : \delta_{\tau(\omega)}(\omega) > \delta\}) > 1 - \varepsilon$. ■

By Lemma 8, and since the payoff processes are uniformly bounded, one obtains the following.

Corollary 9 *Let a stopping time τ and $\varepsilon > 0$ be given. There exists $\delta > 0$ sufficiently small such that for every $\mathcal{F}_\tau$-measurable set $A \subseteq \{\tau < +\infty\}$, and every stopping time μ that satisfies $\tau \leq \mu \leq \tau + \delta$,*

$$|\mathbf{E}_P[a_\mu \mathbf{1}_A] - \mathbf{E}_P[a_\tau \mathbf{1}_A]| \leq 2\varepsilon.$$

3.2 The case $a_t \leq b_t$ for every $t \geq 0$

Definition 10 *Let $\delta > 0$. A mixed strategy ϕ is δ-*pure *if there exists a stopping time μ such that*

$$\phi(r, \cdot) = \mu + r\delta \qquad \forall r \in [0, 1]. \tag{2}$$

Observe that a δ-pure mixed strategy is in particular δ-almost pure. When μ is a stopping time, we sometime denote the δ-pure mixed strategy defined in (2) simply by $\mu + r\delta$.

In this section we prove the following result: when $a_t \leq b_t$ for every $t \geq 0$ the value in mixed strategies exists, it is independent of $(c_t)_{t\geq 0}$, and both players have δ-pure ϵ-optimal strategies, provided δ is sufficiently small.

The idea is the following. Assume player 1 decides to stop at time t. If $c_t \leq a_t$, player 1 wants to mask the exact time in which he stops, so that player 2 cannot stop at the same time. Since payoffs are right-continuous, he can stop randomly in a small interval after time t. If $c_t > a_t$, player 2 prefers that player 1 stops alone at time t rather than to stop simultaneously with player 1 at time t.

Proposition 11 *If $a_t \leq b_t$ for every $t \geq 0$ then the value in mixed strategies exists. Moreover, the value is independent of the process $(c_t)_{t\geq 0}$, and for every $\varepsilon > 0$ there is $\delta_0 > 0$ such that for every $\delta \in (0, \delta_0)$ both players have δ-pure ε-optimal strategies. If $a_t \leq c_t \leq b_t$ for every $t \geq 0$ then the value in pure strategies exists, and there are ϵ-optimal strategies that are independent of $(c_t)_{t\geq 0}$.*

Proof. Consider an auxiliary stopping game $\Gamma^* = (\Omega, \mathcal{A}, P; \mathcal{F}, (a^*_t, b^*_t, c^*_t)_{t\geq 0})$, where $a^*_t = a_t$ and $b^*_t = c^*_t = b_t$ for every $t \geq 0$.

By Theorem 2 the game Γ^* has a value in pure strategies v^*. We will prove that v^* is the value in mixed strategies of the original game. Since Γ^* does not depend on the process $(c_t)_{t\geq 0}$, the second claim in the proposition will follow.

Fix $\varepsilon > 0$. Let μ be an ε-optimal strategy of player 1 in Γ^*. In particular, $\inf_\nu \gamma_{\Gamma^*}(\mu,\nu) \geq v^* - \varepsilon$.

We now construct a mixed strategy ϕ that satisfies $\inf_\nu \gamma_\Gamma(\phi,\nu) \geq v^* - 5\varepsilon$. By Lemma 8 there is $\delta > 0$ such that $p(\{|a_t - a_\mu| < \varepsilon \quad \forall t \in [\mu, \mu+\delta]\}) > 1-\varepsilon$. Define a δ-pure mixed strategy ϕ by

$$\phi(r,\cdot) = \mu + r\delta \qquad \forall r \in [0,1].$$

Let ν be any stopping time. Since μ is ε-optimal in Γ^*, by the definition of Γ^*, and since $\lambda \otimes P(\mu + r\delta = \nu) = 0$,

$$\begin{aligned} v^* - \varepsilon &\leq \gamma_{\Gamma^*}(\mu,\nu) \\ &= \mathbf{E}_P[a_\mu \mathbf{1}_{\{\mu<\nu\}} + b_\nu \mathbf{1}_{\{\mu\geq\nu\}}] \\ &= \mathbf{E}_{\lambda\otimes P}[a_\mu \mathbf{1}_{\{\mu+r\delta<\nu\}} + a_\mu \mathbf{1}_{\{\mu<\nu<\mu+r\delta\}} + b_\nu \mathbf{1}_{\{\mu\geq\nu\}}]. \end{aligned} \tag{3}$$

Since $\lambda \otimes P(\mu + r\delta = \nu) = 0$ and $(c_t)_{t\geq 0}$ is progressively measurable,

$$\begin{aligned} \gamma_\Gamma(\phi,\nu) &= \mathbf{E}_{\lambda\otimes P}\left[a_{\mu+r\delta}\mathbf{1}_{\{\mu+r\delta<\nu\}} + b_\nu \mathbf{1}_{\{\mu+r\delta>\nu\}} + c_\nu \mathbf{1}_{\{\mu+r\delta=\nu<+\infty\}}\right] \\ &= \mathbf{E}_{\lambda\otimes P}\left[a_{\mu+r\delta}\mathbf{1}_{\{\mu+r\delta<\nu\}} + b_\nu \mathbf{1}_{\{\mu+r\delta>\nu\}}\right] \\ &= \mathbf{E}_{\lambda\otimes P}\left[a_{\mu+r\delta}\mathbf{1}_{\{\mu+r\delta<\nu\}} + b_\nu \mathbf{1}_{\{\mu<\nu<\mu+r\delta\}} + b_\nu \mathbf{1}_{\{\mu\geq\nu\}}\right]. \end{aligned} \tag{4}$$

By Corollary 9, and since $a_t \leq b_t$ for every $t \geq 0$,

$$\mathbf{E}_{\lambda\otimes P}[a_\mu \mathbf{1}_{\{\mu<\nu<\mu+r\delta\}}] \leq \mathbf{E}_{\lambda\otimes P}[a_\nu \mathbf{1}_{\{\mu<\nu<\mu+r\delta\}}] + 2\varepsilon \leq \mathbf{E}_{\lambda\otimes P}[b_\nu \mathbf{1}_{\{\mu<\nu<\mu+r\delta\}}] + 2\varepsilon. \tag{5}$$

Corollary 9 implies in addition that

$$\mathbf{E}_{\lambda\otimes P}[a_\mu \mathbf{1}_{\{\mu+r\delta<\nu\}}] \leq \mathbf{E}_{\lambda\otimes P}[a_{\mu+r\delta}\mathbf{1}_{\{\mu+r\delta<\nu\}}] + 2\varepsilon. \tag{6}$$

By (3)-(6),

$$v^* - \varepsilon \leq \gamma_{\Gamma^*}(\mu,\nu) \leq \gamma_\Gamma(\phi,\nu) + 4\varepsilon.$$

Since ν is arbitrary, $\inf_\nu \gamma_\Gamma(\phi,\nu) \geq v^* - 5\varepsilon$.

Consider an auxiliary stopping game $\Gamma^{**} = (\Omega, \mathcal{A}, P; \mathcal{F}, (a_t^{**}, b_t^{**}, c_t^{**})_{t\geq 0})$, where $a_t^{**} = c_t^{**} = a_t$ and $b_t^{**} = b_t$ for every $t \geq 0$.

A symmetric argument to the one provided above proves that the game Γ^{**} has a value v^{**}, and that player 2 has a mixed strategy ψ which satisfies $\sup_\mu \gamma_\Gamma(\mu,\psi) \le v^{**} + 5\varepsilon$.

Since $c_t^{**} = a_t \le b_t = c_t^*$ for every $t \ge 0$, $v^{**} \le v^*$. Since $\sup_\mu \gamma_\Gamma(\mu,\psi) \ge \gamma_\Gamma(\phi,\psi) \ge \inf_\nu \gamma_\Gamma(\phi,\nu)$,

$$v^* \ge v^{**} \ge \sup_\mu \gamma_\Gamma(\mu,\psi) - 5\varepsilon \ge \inf_\nu \gamma_\Gamma(\phi,\nu) - 5\varepsilon \ge v^* - 10\varepsilon.$$

Since ε is arbitrary, $v^* = v^{**}$, so that v^* is the value in mixed strategies of Γ, and ϕ and ψ are 5ε-optimal mixed strategies of the two players.

If $a_t \le c_t \le b_t$ for every $t \ge 0$ then $\gamma_{\Gamma^{**}}(\mu,\nu) \le \gamma_\Gamma(\mu,\nu) \le \gamma_{\Gamma^*}(\mu,\nu)$ for every pair of pure strategies (μ,ν). Hence

$$\begin{aligned} v^{**} &= \sup_\mu \inf_\nu \gamma_{\Gamma^{**}}(\mu,\nu) \le \sup_\mu \inf_\nu \gamma_\Gamma(\mu,\nu) \\ &\le \inf_\nu \sup_\mu \gamma_\Gamma(\mu,\nu) \le \inf_\nu \sup_\mu \gamma_{\Gamma^*}(\mu,\nu) = v^* = v^{**}. \end{aligned}$$

Thus $\sup_\mu \inf_\nu \gamma_\Gamma(\mu,\nu) = \inf_\nu \sup_\mu \gamma_\Gamma(\mu,\nu)$: the value in pure strategies exists. Moreover, any ϵ-optimal strategy of player 1 (resp. player 2) in Γ^* (resp. Γ^{**}) is also ϵ-optimal in Γ. In particular, if $a_t \le c_t \le b_t$ for every $t \ge 0$, both players have ϵ-optimal strategies that are independent of $(c_t)_{t\ge 0}$.
■

3.3 Proof of Theorem 7

Define a stopping time τ by

$$\tau = \inf\{t \ge 0, a_t \ge b_t\},$$

where the infimum of an empty set is $+\infty$. Since $(a_t - b_t)_{t\ge 0}$ is progressively measurable with respect to $(\mathcal{F}_t)_{t\ge 0}$, τ is an $\mathcal{F}$-adapted stopping time (see, e.g., Dellacherie and Meyer, 1975, §IV-50).

The idea is the following. We show that it is optimal for both players to stop at or around time τ (provided the game does not stop before time τ). Hence the problem reduces to the game between times 0 and τ. Since for $t \in [0,\tau[$, $a_t \le b_t$, Proposition 11 can be applied.

The following notation will be useful in the sequel. For a pair of pure strategies (μ, ν), and a set $A \in \mathcal{A}$, we define

$$\gamma_\Gamma(\mu, \nu; A) = \mathbf{E}_P[\mathbf{1}_A(a_\mu \mathbf{1}_{\{\mu<\nu\}} + b_\mu \mathbf{1}_{\{\mu>\nu\}} + c_\mu \mathbf{1}_{\{\mu=\nu<+\infty\}})].$$

This is the expected payoff restricted to A. For a pair of mixed strategies (ϕ, ψ) we define

$$\gamma_\Gamma(\phi, \psi; A) = \int_{[0,1]^2} \gamma_\Gamma(\mu_r, \nu_s; A) dr\ ds,$$

where μ_r and ν_s are the sections of ϕ and ψ respectively.

Set

$$\begin{aligned} A_0 &= \{\tau = +\infty\}, \\ A_1 &= \{\tau < +\infty\} \cap \{c_\tau \geq a_\tau \geq b_\tau\}, \\ A_2 &= \{\tau < +\infty\} \cap \{a_\tau > c_\tau \geq b_\tau\}, \text{ and} \\ A_3 &= \{\tau < +\infty\} \cap \{a_\tau \geq b_\tau > c_\tau\}. \end{aligned}$$

Observe that (A_0, A_1, A_2, A_3) is an $\mathcal{F}_\tau$-measurable partition of Ω.

Define a $\mathcal{F}_\tau$-measurable function w by

$$w = a_\tau \mathbf{1}_{A_1} + c_\tau \mathbf{1}_{A_2} + b_\tau \mathbf{1}_{A_3}.$$

Define a stopping game $\Gamma^* = (\Omega, \mathcal{A}, P, (\mathcal{F}_t)_{t\geq 0}, (a^*_t, b^*_t, c^*_t)_{t\geq 0})$ by:

$$a^*_t = \begin{cases} a_t & t < \tau \\ w & t \geq \tau \end{cases}, \quad b^*_t = \begin{cases} b_t & t < \tau \\ w & t \geq \tau \end{cases}, \quad c^*_t = \begin{cases} c_t & t < \tau \\ w & t \geq \tau \end{cases}.$$

That is, the payoff is set to w at and after time τ.

The game Γ^* satisfies the assumptions of Proposition 11, hence it has a value in mixed strategies V. Moreover, for every $\varepsilon > 0$ both players have δ-pure ε-optimal strategies, provided $\delta > 0$ is sufficiently small.

We now prove that V is the value of the game Γ as well. Fix $\varepsilon > 0$. We only show that player 1 has a mixed strategy ϕ such that $\inf_\nu \gamma_\Gamma(\phi, \nu) \geq V - 7\varepsilon$. An analogous argument shows that player 2 has a mixed strategy

ψ such that $\sup_\mu \gamma_\Gamma(\mu, \psi) \leq V + 7\varepsilon$. Since ε is arbitrary, V is indeed the value in mixed strategies of Γ.

Assume δ is sufficiently small so that the following conditions hold (by the proofs of Lemma 8 and Proposition 11 such δ exists).

(**C1**) Player 1 has a δ-pure ε-optimal strategy $\phi^* = \mu + r\delta$ in Γ^*.

(**C2**) $P(\{\mu + \delta < \tau\}) \geq P(\{\mu < \tau\}) - \varepsilon/M$, where $M \in]0, +\infty[$ is a uniform bound of the payoff processes.

(**C3**) $P(\{|a_t - a_\tau| < \varepsilon, |b_t - b_\tau| < \varepsilon \quad \forall t \in [\tau, \tau + \delta]\}) > 1 - \varepsilon$.

We now claim that one can choose μ so that $\mu \leq \tau$. Indeed, assume that $P(\{\mu > \tau\}) > 0$. The set $\{\mu > \tau\}$ is $\mathcal{F}_\tau$-measurable. Define a stopping time $\mu' = \min\{\mu, \tau\}$. We will prove that the δ-pure strategy $\phi' = \mu' + r\delta$ is also ε-optimal in Γ^*, which establishes the claim. Given a stopping time ν define a stopping time ν' as follows: $\nu' = \tau$ over $\{\mu > \tau\}$, and $\nu' = \nu$ otherwise. Then

$$V - \varepsilon \leq \gamma_{\Gamma^*}(\mu + r\delta, \nu') = \gamma_{\Gamma^*}(\mu + r\delta, \nu'; \{\mu > \tau\}) + \gamma_{\Gamma^*}(\mu + r\delta, \nu'; \{\mu \leq \tau\}).$$

However, $\gamma_{\Gamma^*}(\mu + r\delta, \nu'; \{\mu > \tau\}) = \mathbf{E}_{\lambda \otimes P}[w \mathbf{1}_{\{\mu > \tau\}}] = \gamma_{\Gamma^*}(\mu' + r\delta, \nu; \{\mu > \tau\})$, and since $\mu = \mu'$ and $\nu = \nu'$ over $\{\mu \leq \tau\}$, $\gamma_{\Gamma^*}(\mu + r\delta, \nu'; \{\mu \leq \tau\}) = \gamma_{\Gamma^*}(\mu' + r\delta, \nu; \{\mu \leq \tau\}$. Therefore

$$\gamma_{\Gamma^*}(\mu' + r\delta, \nu) = \gamma_{\Gamma^*}(\mu + r\delta, \nu') \geq V - \varepsilon.$$

Since ν is arbitrary, $\mu' + r\delta$ is ε-optimal, as desired.

Define a mixed strategy ϕ as follows.

$$\phi(r, \cdot) = \begin{cases} \mu + r\delta & \{\mu < \tau\} \cup A_0, \\ \tau & \{\mu = \tau\} \cap (A_1 \cup A_2), \\ \mu + r\delta & \{\mu = \tau\} \cap A_3. \end{cases}$$

Observe that ϕ is δ-almost pure.

The mixed strategies ϕ and ϕ^* differ only over the set $\{\mu = \tau\} \cap (A_1 \cup A_2)$. Since over this set the payoff in Γ^* is w provided the game

terminates after time τ, whatever the players play, ϕ is an ε-optimal mixed strategy in Γ^*.

Let ν be an arbitrary pure strategy of player 2. Define a partition (B_0, B_1, B_2) of $[0,1] \times \Omega$ by

$$\begin{aligned} B_0 &= \{\mu + \delta < \tau\} \cup \{\nu < \tau\}, \\ B_1 &= \{\mu < \tau < \mu + \delta\} \cap \{\nu \geq \tau\}, \text{ and} \\ B_2 &= (\{\mu = \tau \text{ or } \mu = +\infty\}) \cap \{\nu \geq \tau\}. \end{aligned}$$

Over B_0 the game terminates before time τ under (ϕ, ν). In particular,

$$\gamma_\Gamma(\phi, \nu; B_0) = \gamma_{\Gamma^*}(\phi, \nu; B_0). \tag{7}$$

By (**C2**) $\lambda \otimes P(B_1) < \varepsilon/M$, so that

$$\gamma_\Gamma(\phi, \nu; B_1) \geq \gamma_{\Gamma^*}(\phi, \nu; B_1) - 2\varepsilon. \tag{8}$$

Over $B_2 \cap A_0$ the game never terminates under (ϕ, ν), so that

$$\gamma_\Gamma(\phi, \nu; B_2 \cap A_0) = \gamma_{\Gamma^*}(\phi, \nu; B_2 \cap A_0) = 0. \tag{9}$$

Over $A_1 \cup A_2$, $\min\{a_\tau, c_\tau\} \geq w$, so that

$$\begin{aligned} \gamma_\Gamma(\phi, \nu; B_2 \cap (A_1 \cup A_2)) &= \mathbf{E}_{\lambda \otimes P}[\mathbf{1}_{B_2 \cap (A_1 \cup A_2)}(a_\tau \mathbf{1}_{\{\tau < \nu\}} + c_\tau \mathbf{1}_{\{\tau = \nu\}})] \\ &\geq \mathbf{E}_{\lambda \otimes P}[w \mathbf{1}_{\{\tau \leq \nu\} \cap B_2 \cap (A_1 \cup A_2)}] \qquad (10) \\ &= \gamma_{\Gamma^*}(\phi, \nu; B_2 \cap (A_1 \cup A_2)). \end{aligned}$$

Finally, since $\lambda \otimes P(\{\mu + r\delta = \nu\}) = 0$, by Corollary 9, since $(c_t)_{t \geq 0}$ is progressively measurable, and since $a_\tau \geq b_\tau = w$ over A_3,

$$\begin{aligned} \gamma_\Gamma(\phi, \nu; B_2 \cap A_3) &= \mathbf{E}_{\lambda \otimes P}[\mathbf{1}_{B_2 \cap A_3}(a_{\mu + r\delta} \mathbf{1}_{\{\mu + r\delta < \nu\}} + b_\nu \mathbf{1}_{\{\mu + r\delta > \nu\}} + c_\nu \mathbf{1}_{\{\mu + r\delta = \nu\}})] \\ &= \mathbf{E}_{\lambda \otimes P}[\mathbf{1}_{B_2 \cap A_3}(a_{\mu + r\delta} \mathbf{1}_{\{\mu + r\delta < \nu\}} + b_\nu \mathbf{1}_{\{\mu + r\delta > \nu\}})] \\ &\geq \mathbf{E}_{\lambda \otimes P}[\mathbf{1}_{B_2 \cap A_3}(a_\tau \mathbf{1}_{\{\mu + r\delta < \nu\}} + b_\tau \mathbf{1}_{\{\mu + r\delta > \nu\}})] - 4\varepsilon \qquad (11) \\ &\geq \mathbf{E}_{\lambda \otimes P}[w \mathbf{1}_{B_2 \cap A_3}] - 4\varepsilon \\ &= \gamma_{\Gamma^*}(\phi, \nu; B_2 \cap A_3). \end{aligned}$$

Summing Eqs. (7)-(11), and using the ε-optimality of ϕ^* in Γ^*, gives us

$$V - \varepsilon \leq \gamma_{\Gamma^*}(\phi, \nu) \leq \gamma_{\Gamma}(\phi, \nu) + 6\varepsilon,$$

as desired.

4 Extensions

In the present section we construct specific ε-optimal strategies in the spirit of Dynkin (1969) or Rosenberg et al. (2001), and we give conditions for the existence of the value in pure strategies. We then provide two extensions to the basic model.

4.1 Construction of an ε-optimal strategy

Let $\Gamma = (\Omega, \mathcal{A}, P, (\mathcal{F}_t)_{t\geq 0}, (a_t, b_t, c_t)_{t\geq 0})$ satisfy the conditions of Theorem 7.

Define τ, (A_0, A_1, A_2, A_3), w and $\Gamma^* = (\Omega, \mathcal{A}, P, (\mathcal{F}_t)_{t\geq 0}, (a_t^*, b_t^*, c_t^*)_{t\geq 0})$ as in the proof of Theorem 7.

For any stopping time σ let $\Gamma_\sigma^* = (\Omega, \mathcal{A}, P, (\mathcal{F}_t)_{t\geq 0}, (a_t^*, b_t^*, c_t^*)_{t\geq 0})$ be the game starting at time σ; that is, players are restricted to choose strategies that stop with probability 1 at or after time σ.

Lepeltier and Mainguenau (1984, Theorem 13) and Proposition 11 show that this game has a value in mixed strategies X_σ^*. Moreover, the value is independent of $(c_t^*)_{t\geq 0}$.

Using a general result of Dellacherie and Lenglart (1982), Lepeltier and Mainguenau (1984, Theorem 7) show the existence of a right-continuous process $(V_t^*)_{t\geq 0}$ such that $V_\sigma^* = X_\sigma^*$ for every stopping time σ.

For every $\epsilon > 0$ define a stopping time

$$\mu_\varepsilon^* = \inf\left\{t \geq 0 : V_t^* \leq a_t^* + \frac{\varepsilon}{35}\right\}.$$

By definition of τ and Γ^*, one has $\mu_\varepsilon^* \leq \tau$.

Lepeltier and Mainguenau (1984, Theorem 13) and Proposition 11 imply that μ_ε^* is $\frac{\varepsilon}{35}$-optimal for Player 1 in any game $\widetilde{\Gamma} = (\Omega, \mathcal{A}, P, (\mathcal{F}_t)_{t\geq 0}, (a_t^*, b_t^*, d_t)_{t\geq 0})$, where $(d_t)_{t\geq 0}$ is any process satisfying $a_t^* \leq d_t \leq b_t^*$ for any $t \geq 0$.

Let δ be such that $P(\{|a_t - a_{\mu_\varepsilon^*}| < \frac{\varepsilon}{35},\ \forall t \in [\mu_\varepsilon^*, \mu_\varepsilon^* + \delta]\}) > 1 - \frac{\varepsilon}{35}$. From the proof of Proposition 11 we deduce that $\mu_\varepsilon^* + r\delta$ is $\frac{\varepsilon}{7}$-optimal for player 1 in Γ^*.

Now, assume that δ is sufficiently small so that

- $P(\{\mu_\varepsilon^* + \delta < \tau\}) \geq P(\{\mu_\varepsilon^* < \tau\}) - \frac{\varepsilon}{7M}$, and
- $P(\{|a_t - a_\tau| < \frac{\varepsilon}{7}, |b_t - b_\tau| < \frac{\varepsilon}{7} \quad \forall t \in [\tau, \tau + \delta]\}) > 1 - \frac{\varepsilon}{7}$.

Define a mixed strategy ϕ_ε as follows.

$$\phi_\varepsilon(r, \cdot) = \begin{cases} \mu_\varepsilon^* + r\delta & \{\mu_\varepsilon^* < \tau\} \cup A_0, \\ \tau & \{\mu_\varepsilon^* = \tau\} \cap (A_1 \cup A_2), \\ \mu_\varepsilon^* + r\delta & \{\mu_\varepsilon^* = \tau\} \cap A_3. \end{cases}$$

The proof of Theorem 7 implies that ϕ_ε is ε-optimal for player 1 in Γ.

Assume that $c_\tau \geq b_\tau$ a.s. (or, equivalently, that $A_1 \cup A_2 = \Omega$). By the proof of Proposition 11, it is 0-optimal for Player 1 in Γ to stop at time τ, provided the game reaches time τ. If in addition one has $a_t \leq c_t \leq b_t$ for every $t \in [0, \tau[$, by the proof of Proposition 11 and Lepeltier and Mainguenau (1984, Theorem 13) we deduce that the pure stopping time $\inf\{t \geq 0 : V_t^* \leq a_t^* + \varepsilon\}$ is ε-optimal for Player 1 in Γ. Hence one obtains the following.

Proposition 12 *If $c_t \in \mathrm{co}\{a_t, b_t\}$ for every $t \in [0, \tau]$ then the value exists in pure strategies. An ε-optimal strategy for Player 1 is* $\inf\{t \geq 0 : V_t^* \leq a_t^* + \varepsilon\}$, *and an ε-optimal strategy for Player 2 is* $\inf\{t : V_t^* \geq b_t^* - \varepsilon\}$.

Corollary 13 *Every stopping game such that $(a_t, b_t, c_t)_{t \geq 0}$ are continuous and satisfies $c_0 \in \mathrm{co}\{a_0, b_0\}$ admits a value in pure strategies.*

4.2 On final payoff

Our convention is that the payoff is 0 if no player ever stops. In fact, one can add a final payoff as follows. Let χ be an $\mathcal{A}$-measurable and integrable

function. The expected payoff that corresponds to a pair of pure strategies (μ, ν) is:

$$\mathbf{E}_P[a_\mu \mathbf{1}_{\{\mu<\nu\}} + b_\nu \mathbf{1}_{\{\mu>\nu\}} + c_\mu \mathbf{1}_{\{\mu=\nu<+\infty\}} + \chi \mathbf{1}_{\{\mu=\nu=+\infty\}}].$$

The expected payoff can be written as:

$$\begin{aligned}\mathbf{E}_P[\chi] + \mathbf{E}_P\Big[\Big(a_\mu - \mathbf{E}_P^{\mathcal{F}_\mu}[\chi]\Big)\mathbf{1}_{\{\mu<\nu\}} &+ \Big(b_\nu - \mathbf{E}_P^{\mathcal{F}_\nu}[\chi]\Big)\mathbf{1}_{\{\mu>\nu\}} \\ &+ \Big(c_\mu - \mathbf{E}_P^{\mathcal{F}_\mu}[\chi]\Big)\mathbf{1}_{\{\mu=\nu<+\infty\}}\Big],\end{aligned}$$

where $\mathbf{E}_P^{\mathcal{F}_\mu}[\chi]$ is the conditional expectation of χ given the σ-algebra $\mathcal{F}_\mu$.

Define a process $d_t := \mathbf{E}_P^{\mathcal{F}_t}[\chi]$. Since the filtration satisfies the "usual conditions", $(d_t)_{t\geq 0}$ is a right-continuous martingale (see, e.g., Dellacherie and Meyer, 1980, §VI-4). Hence we are reduced to the study of the standard stopping game $\Gamma^* = (\Omega, \mathcal{A}, P, (\mathcal{F}_t)_{t\geq 0}, (a_t^*, b_t^*, c_t^*)_{t\geq 0})$ with $a_t^* = b_t - d_t$, $b_t^* = b_t - d_t$ and $c_t^* = c_t - d_t$.

4.3 On cumulative payoff

In our definition, players receive no payoff before the game stops. One can add a cumulative payoff as follows. Let $(x_t)_{t\geq 0}$ be a progressively measurable process satisfying $\mathbf{E}_P\left[\int_0^{+\infty} |x_t|\, dt\right] < +\infty$, and suppose that the expected payoff that corresponds to a pair of pure strategies (μ, ν) is given by

$$\mathbf{E}_P\left[a_\mu \mathbf{1}_{\{\mu<\nu\}} + b_\nu \mathbf{1}_{\{\mu>\nu\}} + c_\mu \mathbf{1}_{\{\mu=\nu<+\infty\}} + \int_0^{\min\{\mu,\nu\}} x_t dt\right].$$

The expected payoff can be written as

$$\begin{aligned}\mathbf{E}_P\Bigg[\left(a_\mu + \int_0^\mu x_t dt\right)\mathbf{1}_{\{\mu<\nu\}} + \left(b_\nu + \int_0^\nu x_t dt\right)\mathbf{1}_{\{\mu>\nu\}} \\ + \left(c_\mu + \int_0^\mu x_t dt\right)\mathbf{1}_{\{\mu=\nu<+\infty\}}\Bigg] + \mathbf{E}_P\left[\mathbf{1}_{\{\mu=\nu=+\infty\}} \times \int_0^\infty x_t dt\right].\end{aligned}$$

Thus, the game is equivalent to the stopping game $\Gamma^* = (\Omega, \mathcal{A}, P, (\mathcal{F}_t)_{t\geq 0}, (a_t^*, b_t^*, c_t^*)_{t\geq 0})$ with terminal payoff $\chi = \int_0^\infty x_t dt$, where $a_t^* = a_t + \int_0^t x_s ds$, $b_t^* = b_t + \int_0^t x_s ds$, and $c_t^* = c_t + \int_0^t x_s ds$.

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